Week 09: System Analysis in the Time Domain

Mahmut Selman Sakar

Institute of Mechanical Engineering, EPFL

Announcements

- No new problem set this week
- PS8 contains questions on this week's lecture
- Try to solve all the exercises from polycopie (Chapter 6)

First Order Systems

- Systems with a transfer function that has a characteristic polynomial of degree one

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}$$

The pole of the system is at $p = -\frac{1}{\tau}$

Time constant

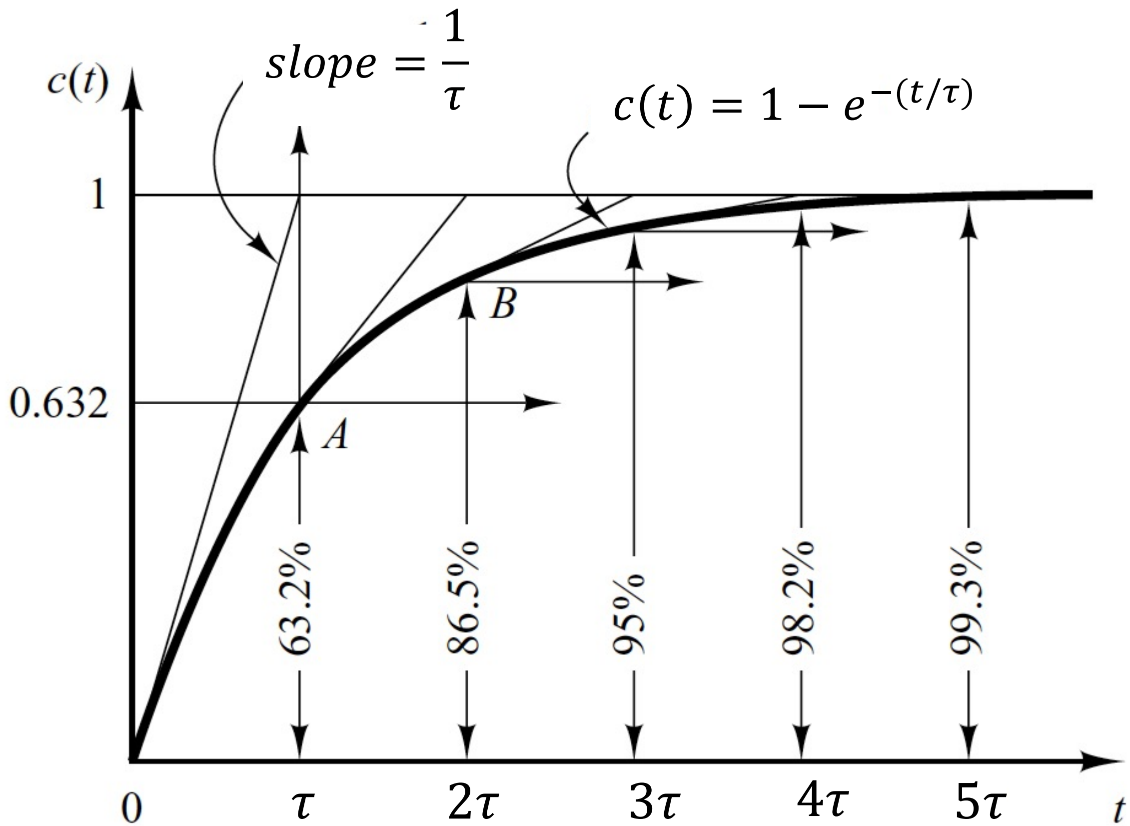
<u>Steady State Gain</u>

$$K = \lim_{s \to 0} G(s)$$

First Order Systems: Step Response

$$c(t) = \frac{y(t)}{KA}$$

Normalized response curve



Second Order Systems

- Transfer function without zeros

$$G(s) = \frac{Y(s)}{U(s)} = K \frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}$$

K : Steady-state output (DC Gain)
ζ : Damping ratio
ω_0 : Undamped natural frequency

- For now, assume that $\zeta \geq 0$

Second Order Systems

- Transfer function without zeros

$$G(s) = \frac{Y(s)}{U(s)} = K\,\frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}$$

- **Poles of the System**

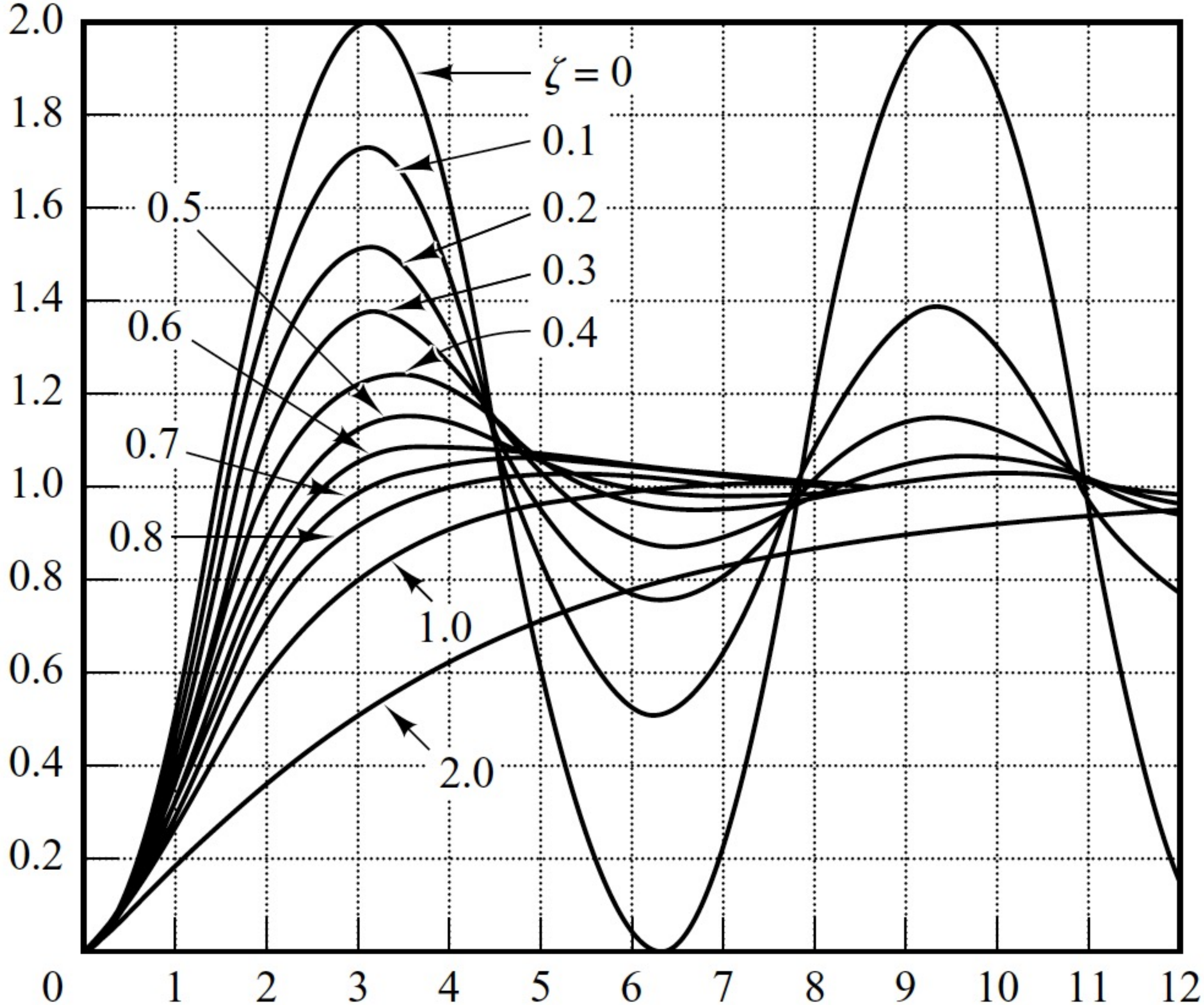
$$p_{1,2} = -\omega_0\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

- Poles are either
  - distinct real number,
  - repeated real numbers, or
  - complex conjugates

Second Order Systems

- Overdamped response (real and distinct poles)
- Critically damped response (real and repeated poles)
- Underdamped response (complex conjugate poles)
- Undamped response (complex conjugate poles without real parts)

Step Response of Second Order Systems



Lecture Overview

- Underdamped Response
- Stability
- Higher order systems

Second Order Systems

- **Underdamped Case (Poles are complex conjugates)**

$$0 \leq \zeta < 1$$

$$G(s) = \frac{Y(s)}{U(s)} = K\frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2} = K\frac{a^2 + \bar{\omega}^2}{(s+a)^2 + \bar{\omega}^2}$$

Damped natural frequency

$$\bar{\omega} = \omega_0\sqrt{1-\zeta^2}$$

Attenuation

$$a = \zeta\omega_0$$

Second Order Systems

- **Underdamped Case (Poles are complex conjugates)** $0 \leq \zeta < 1$

$$G(s) = \frac{Y(s)}{U(s)} = K\frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2} = K\frac{a^2 + \bar{\omega}^2}{(s+a)^2 + \bar{\omega}^2}$$

- **Step Response**

$$u(t) = A\varepsilon(t) \qquad Y(s) = \frac{1}{s} - \frac{s + \zeta\omega_0}{(s+\zeta\omega_0)^2 + \bar{\omega}^2} - \frac{\zeta\omega_0}{(s+\zeta\omega_0)^2 + \bar{\omega}^2}$$

$$y(t) = \varepsilon(t)KA\left\{1 - e^{-at}\left(\cos\bar{\omega}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\bar{\omega}t\right)\right\}$$

Transient Response (Underdamped System)

- **Delay time, t_d:** Time required for the response to reach half of the final value the very first time

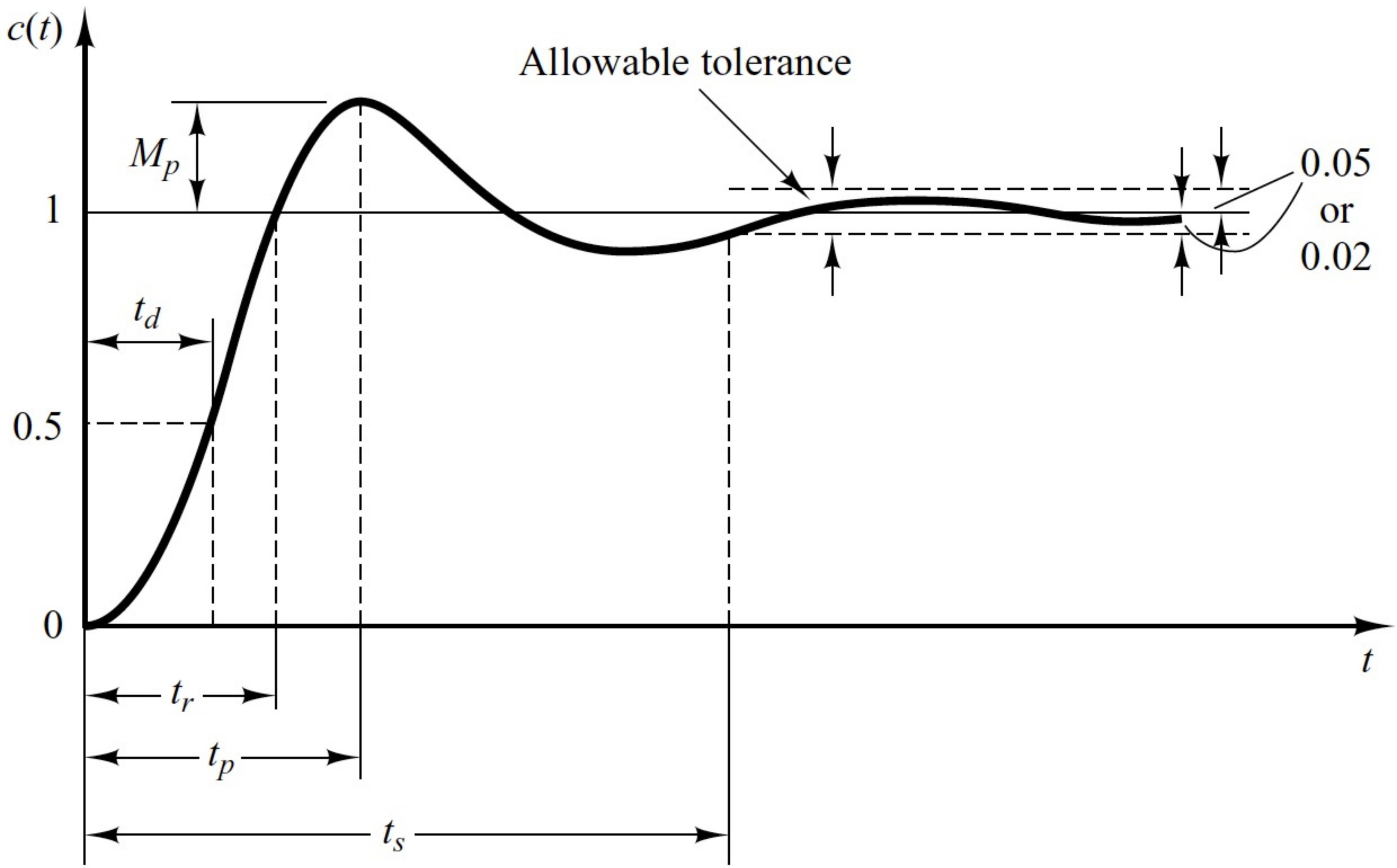
- **Rise time, t_t:** Time required for the response to rise from 0% to 100% (underdamped system) or from 10% to 90% (overdamped system)

- **Peak time, t_p:** Time required for the response to reach the first peak of the overshoot.

- **Maximum percent overshoot, M_p**

- **Settling time, t_s:** Time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%)

Transient Response

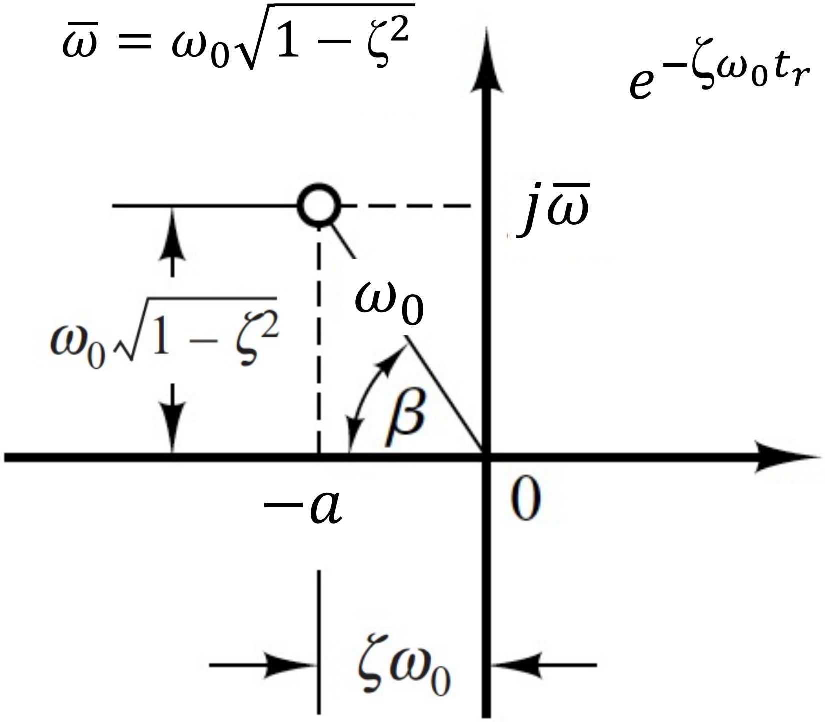


Rise Time

$$c(t_r) = 1 - e^{-\zeta\omega_0 t_r}\left(\cos\bar{\omega}t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\bar{\omega}t_r\right) = 1$$

$$\bar{\omega} = \omega_0\sqrt{1-\zeta^2}$$

$$e^{-\zeta\omega_0 t_r} \neq 0 \Rightarrow \cos\bar{\omega}t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\bar{\omega}t_r = 0$$



$$\tan\bar{\omega}t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\bar{\omega}}{a}$$

$$t_r = \frac{1}{\bar{\omega}}\tan^{-1}\left(\frac{\bar{\omega}}{-a}\right) = \frac{\pi-\beta}{\bar{\omega}}$$

$$p_{1,2} = -\omega_0(\zeta \pm j\sqrt{1-\zeta^2})$$

Peak Time

$$\frac{dc}{dt} = \zeta\omega_0 e^{-\zeta\omega_0 t_r}\left(\cos\bar{\omega}t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\bar{\omega}t_r\right) + e^{-\zeta\omega_0 t_r}\left(\bar{\omega}\sin\bar{\omega}t_r - \frac{\zeta\bar{\omega}}{\sqrt{1-\zeta^2}}\cos\bar{\omega}t_r\right)$$

$$\left.\frac{dc}{dt}\right|_{t=t_p} = \left(\sin\bar{\omega}t_p\right)\frac{\omega_0}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t_p} = 0$$

$\sin\bar{\omega}t_p = 0$ ⇨ $\bar{\omega}t_p = 0, \pi, 2\pi, 3\pi, \ldots$

First peak overshoot: $\bar{\omega}t_p = \pi$ ⇨ $t_p = \dfrac{\pi}{\bar{\omega}}$

Maximum Overshoot

- Maximum overshoot occurs at the peak time

$$M_p = c(t_p) - 1 = -e^{-\zeta\omega_0(\pi/\overline{\omega})}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$= e^{-(a\,/\overline{\omega})\pi}$$

- Maximum percent overshoot

$$\text{OS\%} = e^{-\zeta\pi/\sqrt{1-\zeta^2}}\times 100\%$$

Settling Time

$$c(t) = 1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}} \sin\left(\bar{\omega}t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

- **Envelope Curves**

$$1 \pm \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}$$

- **Time Constant**

$$\tau = \frac{1}{\zeta\omega_0} = \frac{1}{a}$$

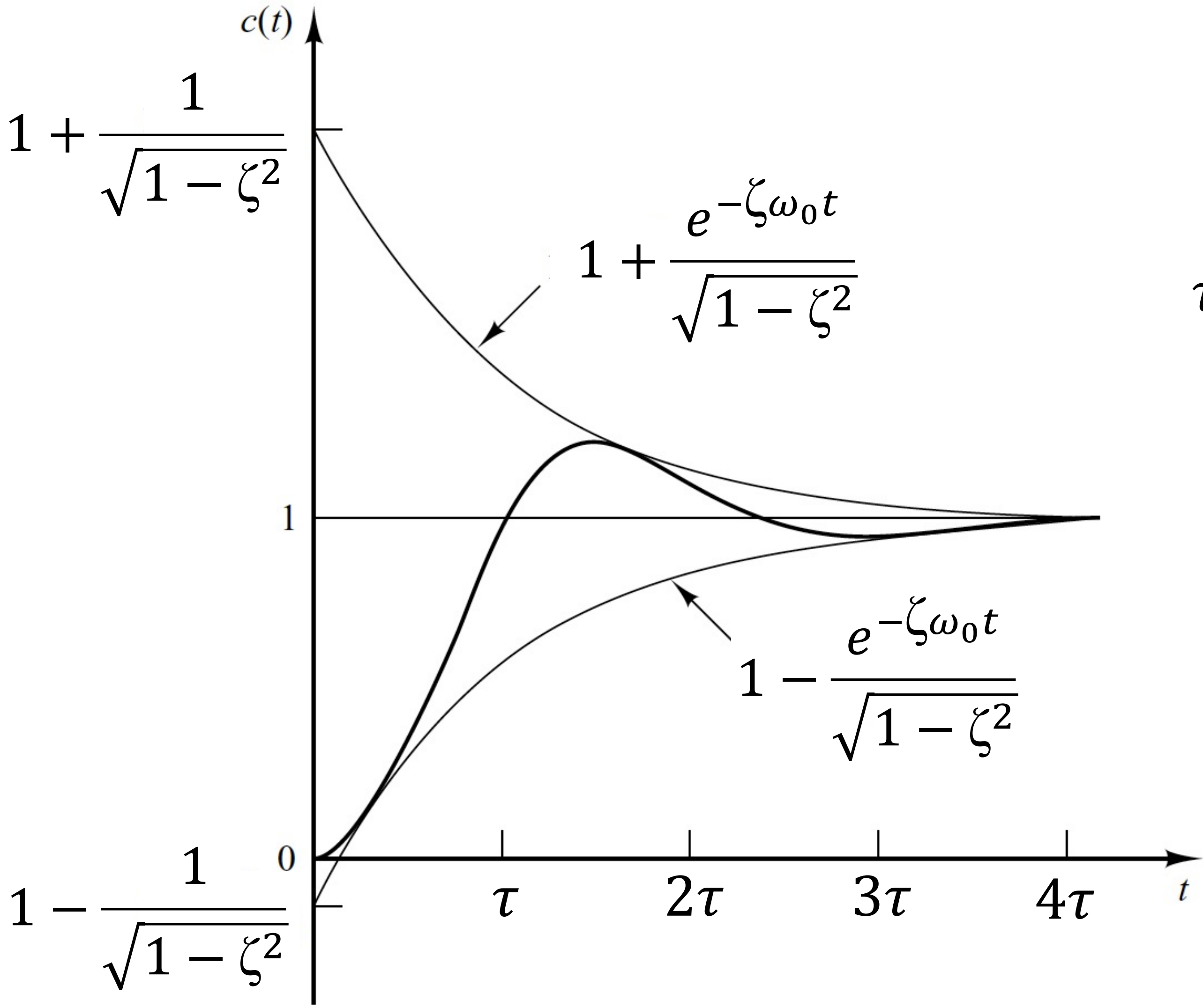
- **2% Criterion**

$$t_s = 4\tau = \frac{4}{a} = \frac{4}{\zeta\omega_0}$$

- **5% Criterion**

$$t_s = 3\tau = \frac{3}{a} = \frac{3}{\zeta\omega_0}$$

Settling Time



$$\tau = \frac{1}{\zeta\omega_0}$$

Comments on Transient-Response Specifications

- Small values of damping coefficient (that is, $\zeta < 0.4$)
  - Excessive overshoot in the transient response

- Large values of damping coefficient (that is, $\zeta > 0.8$)
  - Sluggish response

- If we determine damping coefficient according to permissible maximum overshoot, settling time will be primarily determined by undamped natural frequency.

- Rapid response requires large natural frequency

- Overdamped systems have large settling time

Comparison of Different Second Order Systems

- If the poles have the same real part
  - Same time constant
  - Same settling time

- If the poles have the same imaginary part
  - Same period of oscillation
  - Same peak time

- If the poles have the same damping coefficient
  - Same overshoot

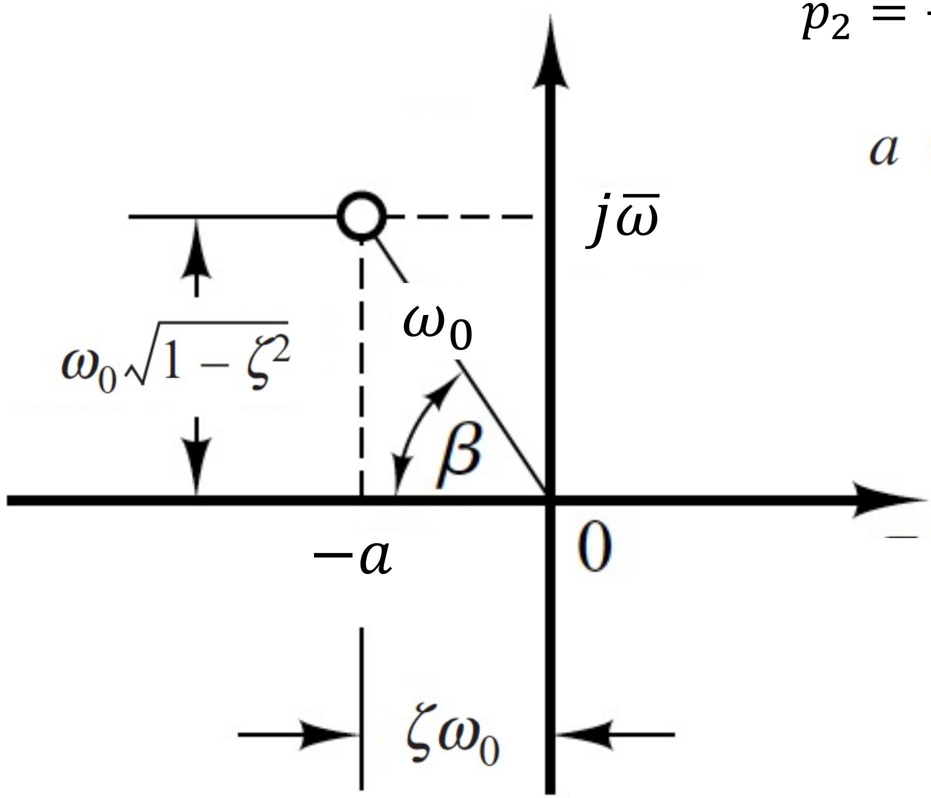
$$p_{1,2} = -\omega_0\left(\zeta \pm j\sqrt{1-\zeta^2}\right) = -a \pm j\bar{\omega}$$

Graphical Representation

$$p_{1,2} = -\omega_0(\zeta \pm j\sqrt{1-\zeta^2})$$

$$p_1 = -\omega_0\zeta - j\omega_0\sqrt{1-\zeta^2} = -a - j\bar{\omega}$$

$$p_2 = -\omega_0\zeta + j\omega_0\sqrt{1-\zeta^2} = -a + j\bar{\omega}$$



$$a = \zeta\omega_0 \qquad \bar{\omega} = \omega_0\sqrt{1-\zeta^2}$$

$$t_r = \frac{\pi - \beta}{\bar{\omega}} \qquad t_p = \frac{\pi}{\bar{\omega}}$$

$$OS\% = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$$

$$t_s = \frac{3}{a}$$ 5% criterion

Example

ω_n is ω_0

$$G(s) = \frac{K}{s^2 + 4s + K} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- For any gain K:
  - $\zeta\omega_n = 2$ rad/s for any choice of gain K
  - Settling time: $T_s = \frac{4}{\zeta\omega_n} = 2$ s
- For $K = 16, \omega_n = \sqrt{16} = 4\frac{rad}{s}$, $\zeta = \frac{2}{\omega_n} = 0.5$
- Increasing K
  (i) decreases ζ (increases OS%)
  (ii) increases ω_n and ω_d (decreases T_p) : faster response

Poles, Zeros and System Properties

- The homogenous (free, natural) response can be written as

$$y_h(t) = \sum_{i=1}^{n} C_i e^{p_i t}$$

- The transfer function poles are the roots of the characteristic equation, and also the eigenvalues of the system **A** matrix.

$$\dot{x} = \mathrm{A}x$$

Poles, Zeros and System Properties

- A real pole in the left-half of the s-plane defines an exponential decaying component in the homogenous response.

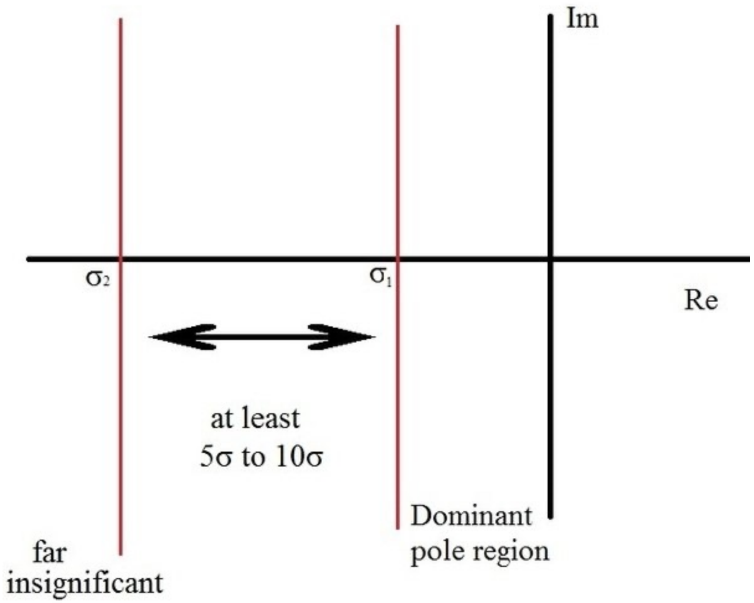
$$p_i = -a \qquad y(t) = Ce^{-at}$$

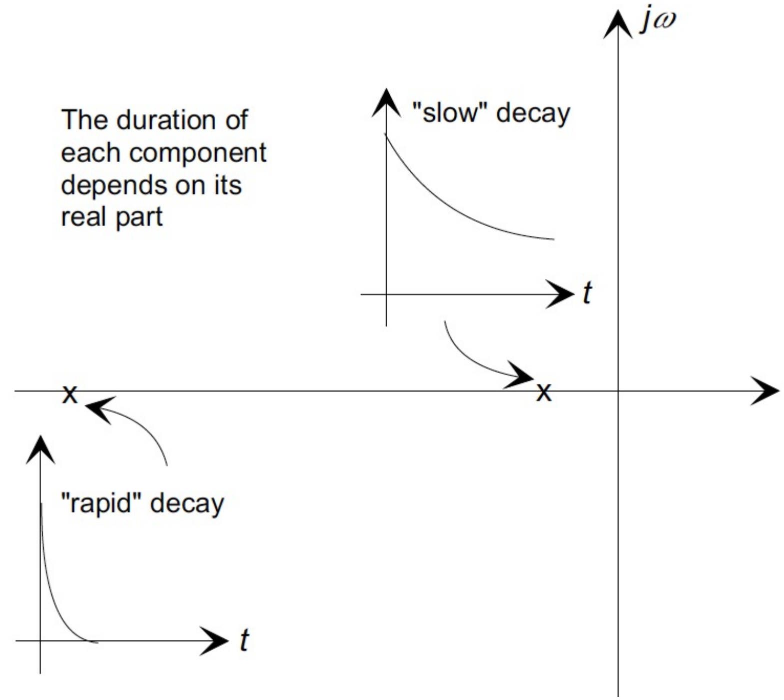
- The rate of decay is determined by the pole location
  - poles far away from the origin in the left-half plane correspond to components that decay rapidly.

- Dominant pole approximation
  - If the system has a cluster of poles and zeros that are much closer (5 times or more) to the origin than the other poles and zeros, the system can be approximated by a lower order system with only those dominant poles and zeros.

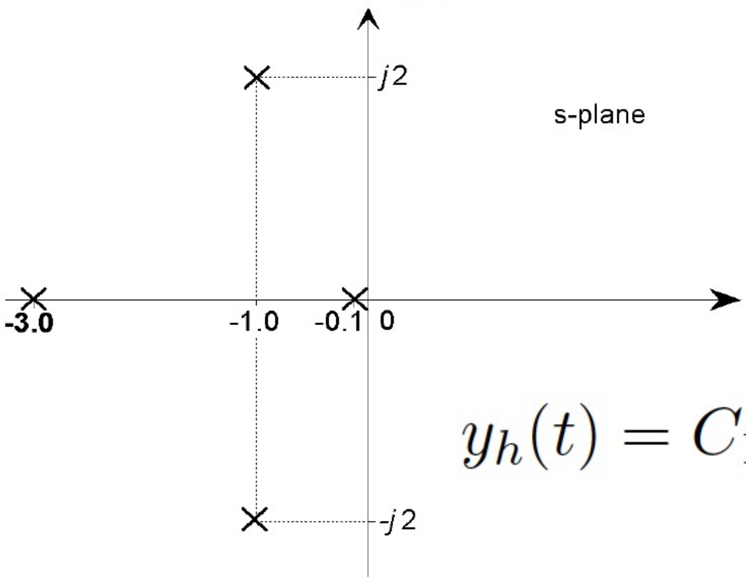
Dominant Pole Approximation

- One of the two decaying exponentials decreases much faster than the other
- Faster decaying exponential term may be neglected (smaller time constant)
- Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system.





Example



$$y_h(t) = C_1 e^{-3t} + C_2 e^{-0.1t} + Ae^{-t} \sin(2t + \phi)$$

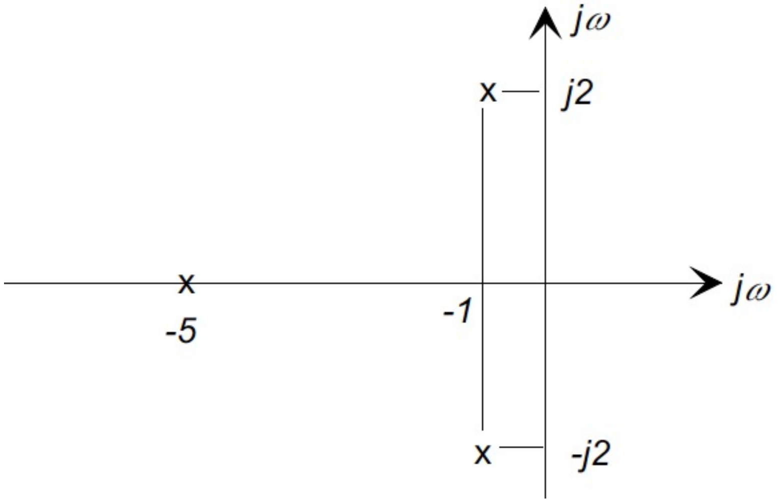
The term e^{-3t}, with a time-constant τ of 0.33 seconds, decays rapidly and is significant only for approximately 4τ or 1.33seconds.

The response has an oscillatory component $Ae^{-t}\sin(2t + \phi)$ defined by the complex conjugate pair, and exhibits some overshoot. The oscillation will decay in approximately four seconds because of the e^{-t} damping term.

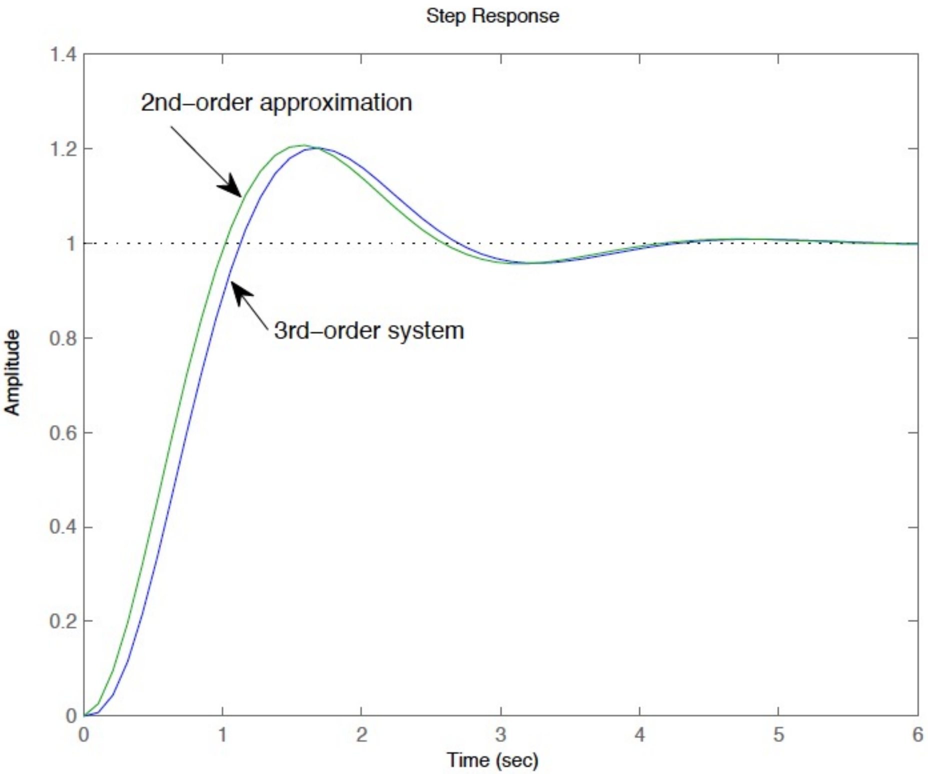
The term $e^{-0.1t}$, with a time-constant $\tau = 10$ seconds, persists for approximately 40 seconds. It is therefore the *dominant* long term response component in the overall homogeneous response.

Example

$$G(s) = \frac{50}{s^3 + 12s^2 + 255s + 50} = \frac{50}{(s+10)(s^2+2s+5)}$$



$$G(s) \approx \frac{5}{s^2 + 2s + 5}$$



Poles, Zeros and System Properties

- A real pole in the right-half plane corresponds to an exponentially increasing component in the homogenous response, thus defining the system to be **unstable**.

$$p_i = a \qquad y(t) = Ce^{at}$$

- In a stable system all components of the homogenous response must decay to zero as time increases.

- If any pole has a **positive real part** there is a component in the output that increases without bound, causing the system to be **unstable**.

- For an LTI system to be stable, all of its poles must have negative real parts, that is they must all lie within the left-half of the s-plane.

Stability

- Stability of an LTI system does not depend on the input function
- The poles of the input contribute only to steady-state response

Poles, Zeros and System Properties

- A complex conjugate pole pair in the left half of the s-plane combine to generate a response component that is decaying sinusoid.

$$p_i = \sigma \pm j\omega = -a \pm j\omega \qquad y(t) = Ae^{-at}\sin(\omega t + \phi)$$

- The rate of decay is specified by the real component of the pole, the frequency of oscillation is determined by the imaginary component of the pole.

Poles, Zeros and System Properties

- An imaginary pole pair, that is a pole pair lying on the imaginary axis, generates oscillatory component with a constant amplitude determined by initial conditions.

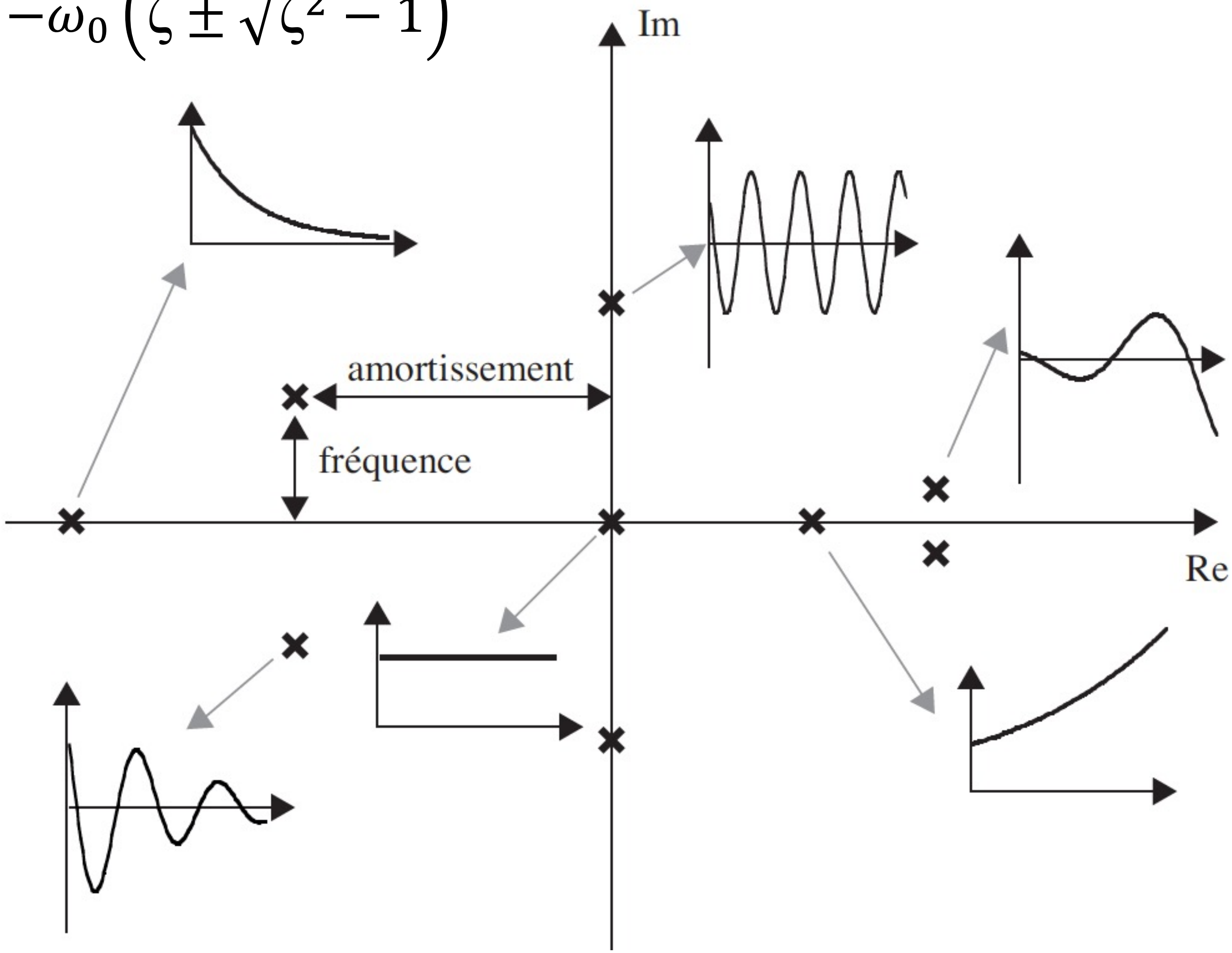
$$p_i = j\omega \qquad y(t) = A\sin(\omega t + \phi)$$

- Mathematically, closed-loop poles on the jw axis will yield oscillations, the amplitude of which is neither decaying nor growing with time. In practical cases, where noise is present, however, the amplitude of oscillations may increase at a rate determined by the noise power level. Therefore, a control system should not have closed-loop poles on the jw axis.

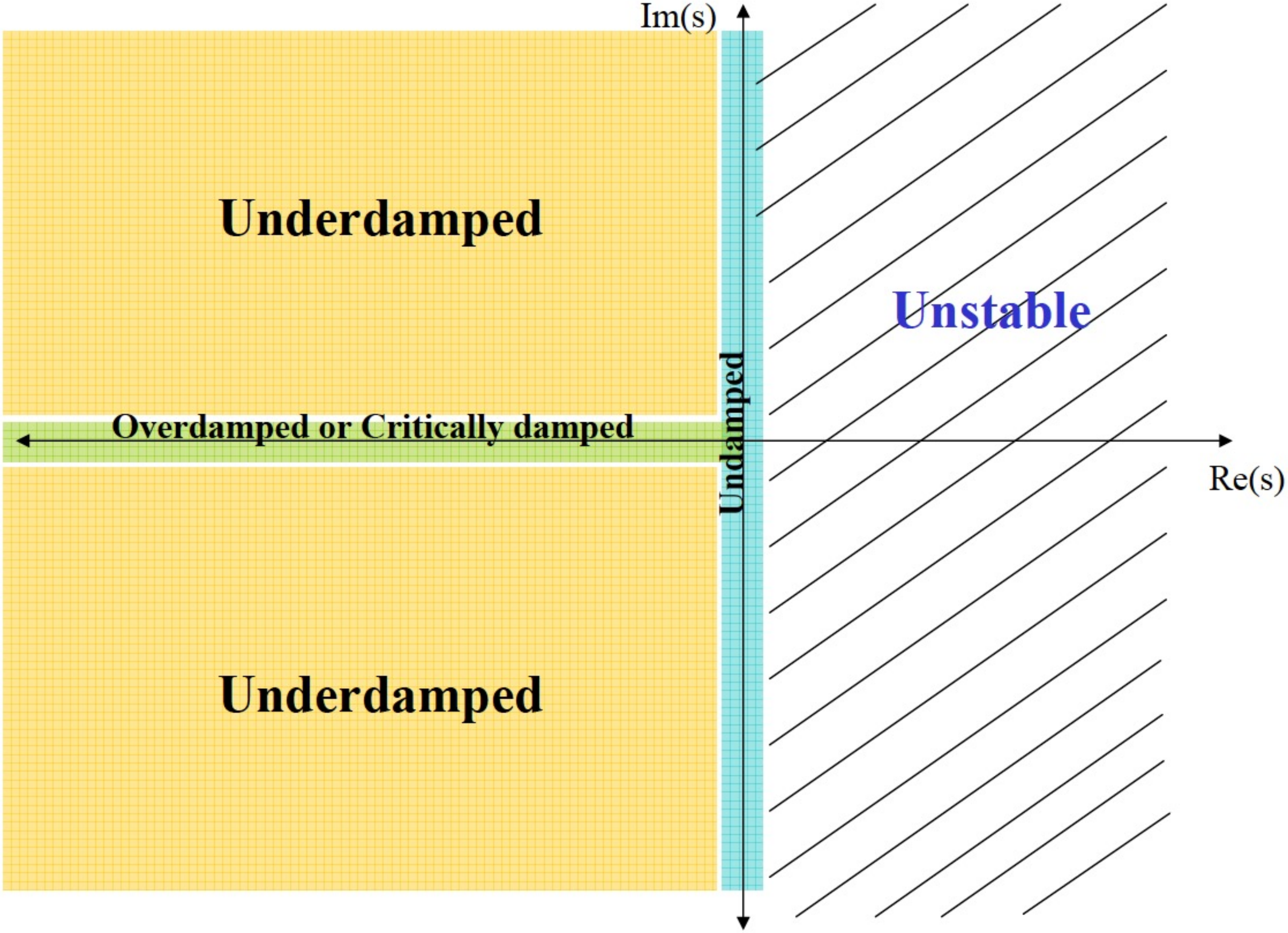
- A complex pole pair in the right half plane generates an exponentially increasing component (unstable system).

Poles and Stability

$$p_{1,2} = -\omega_0\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)$$



Poles and Stability



Effect of Zeros on Time Response

- The magnitudes of the residues depend on both the poles and zeros
- Faster response
- Increased overshoot

$$G(s) = \frac{Y(s)}{U(s)} = K \frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}$$

Effect of Zeros on Time Response

$$G_1(s) = \frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2} \qquad G_2(s) = \frac{\left(\frac{1}{z}s + 1\right){\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}$$

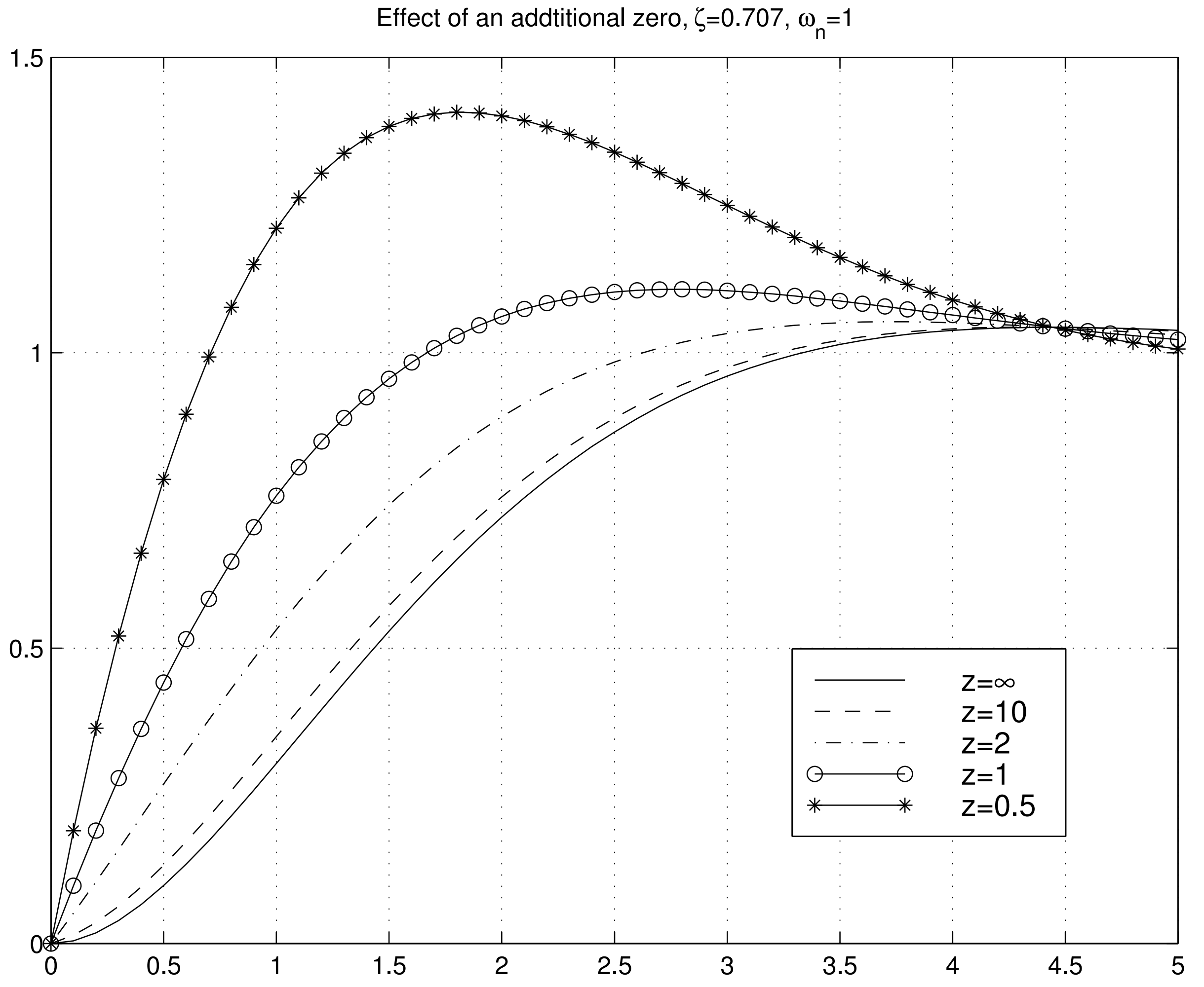
$$G_2(s) = \frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2} + \frac{1}{z}s\frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2} = G_1(s) + \frac{1}{z}sG_1(s)$$

$$Y_2(s) = \left(G_1(s) + \frac{1}{z}sG_1(s)\right)\frac{1}{s} = Y_1(s) + \frac{1}{z}sY_1(s)$$

$$y_2(t) = y_1(t) + \frac{1}{z}\dot{y}_1(t)$$

The step response of the second order system with a zero at s = -z is given by the step response of the original system plus a scaled version of the derivative of the step response of the original system.

Effect of Zeros on Time Response



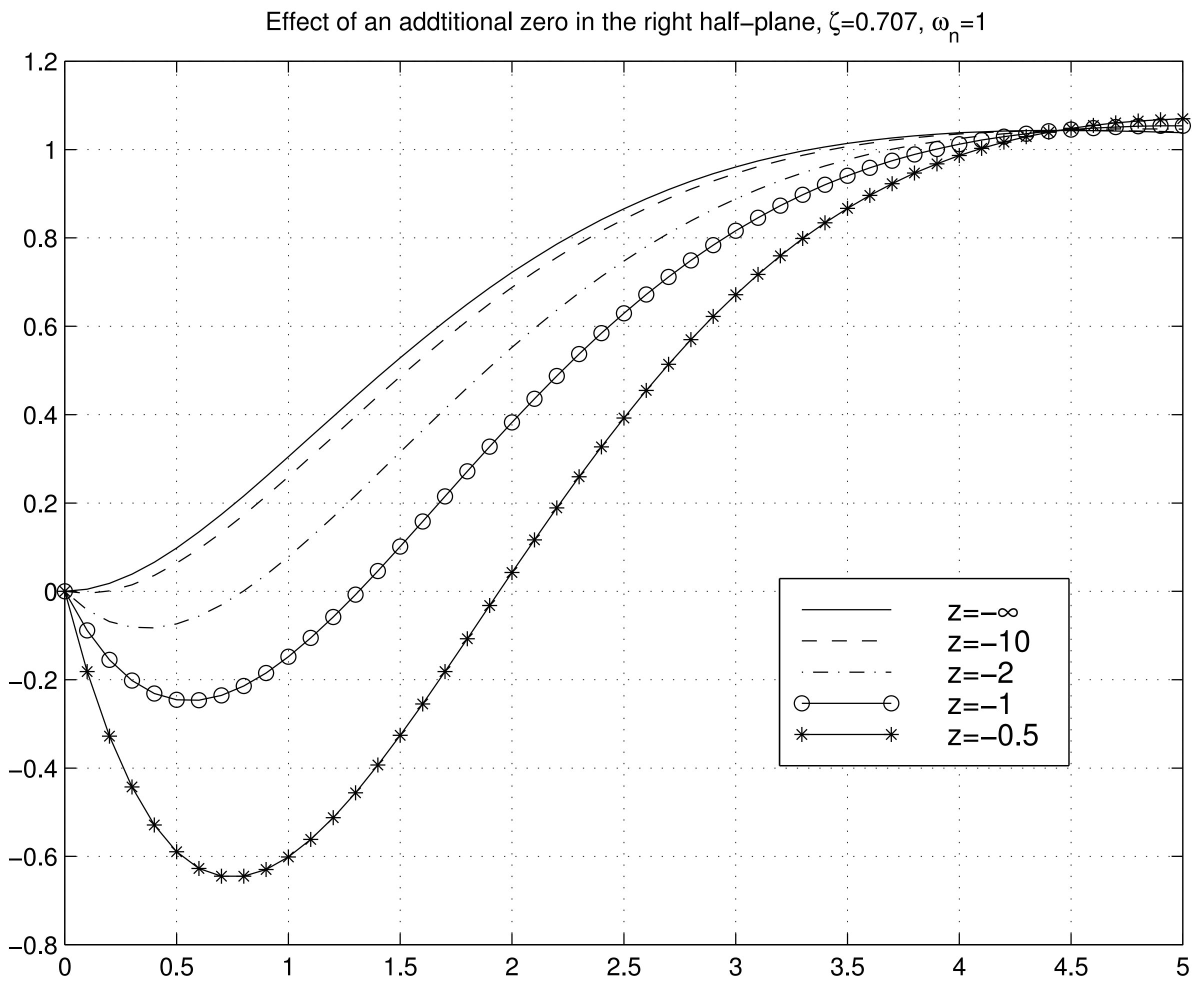
Effect of Zeros on Time Response

- Note that as z increases (i.e., as the zero moves further into the left half plane), the term $1/z$ becomes smaller, and thus the contribution of the term dy_1/dt decreases (i.e., the step response of this system starts to resemble the step response of the original system).
- The effect of a left-half plane zero is to increase the overshoot, decrease the peak time, and decrease the rise time; the settling time is not affected too much. In other words, a left-half plane zero makes the step response faster.

Effect of Zeros on Time Response

- If z is negative (which corresponds to the zero being in the right half plane), the derivative dy_1/dt is subtracted from $y_1(t)$ to produce the output $y_2(t)$.
- The response becomes slower
- No change in stability
- Note that the response can actually go in the opposite direction before rising to 1. This phenomenon is called undershoot.

Effect of Zeros on Time Response



Pole/zero cancellation

- For any system in which one or more poles have been canceled by zeros in the transfer function, the modal components e^{pt} corresponding to the canceled poles will not appear in the output for any input u(t)

- We say that the canceled modes are not excited by the input

- As a zero approaches a pole, the amplitude of the modal component e^{pt} corresponding to the pole decreases, for any input u(t).

– Example
  – Poles at $p_{1,2} = -5 \pm j5$, $p_3 = -4$ and zero at $z_1 = -4.05$

Example

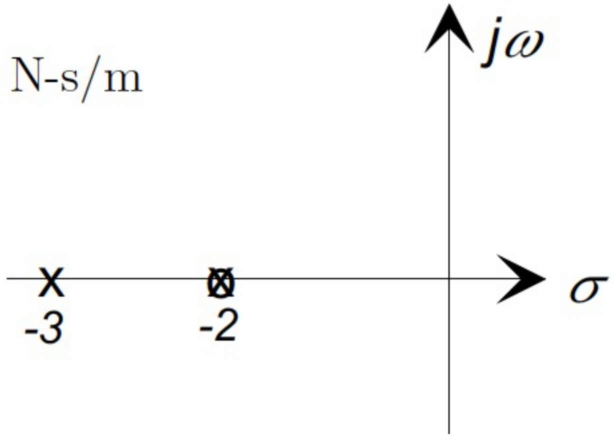
- Consider the following system

$$G(s) = \frac{v_m(s)}{V_s(s)} = \frac{B_1 s + K}{ms^2 + (B_1 + B_2)s + K}$$

$m_1 = 1/3$ kg, $K = 2$ N/m, $B_1 = 1$ N-s/m, $B_2 = 2/3$ N-s/m

$$G(s) = \frac{s+2}{(1/3)s^2 + (5/3)s + 2} = \frac{3(s+2)}{(s+2)(s+3)}$$

$$G(s) = \frac{3}{s+3}$$



Example

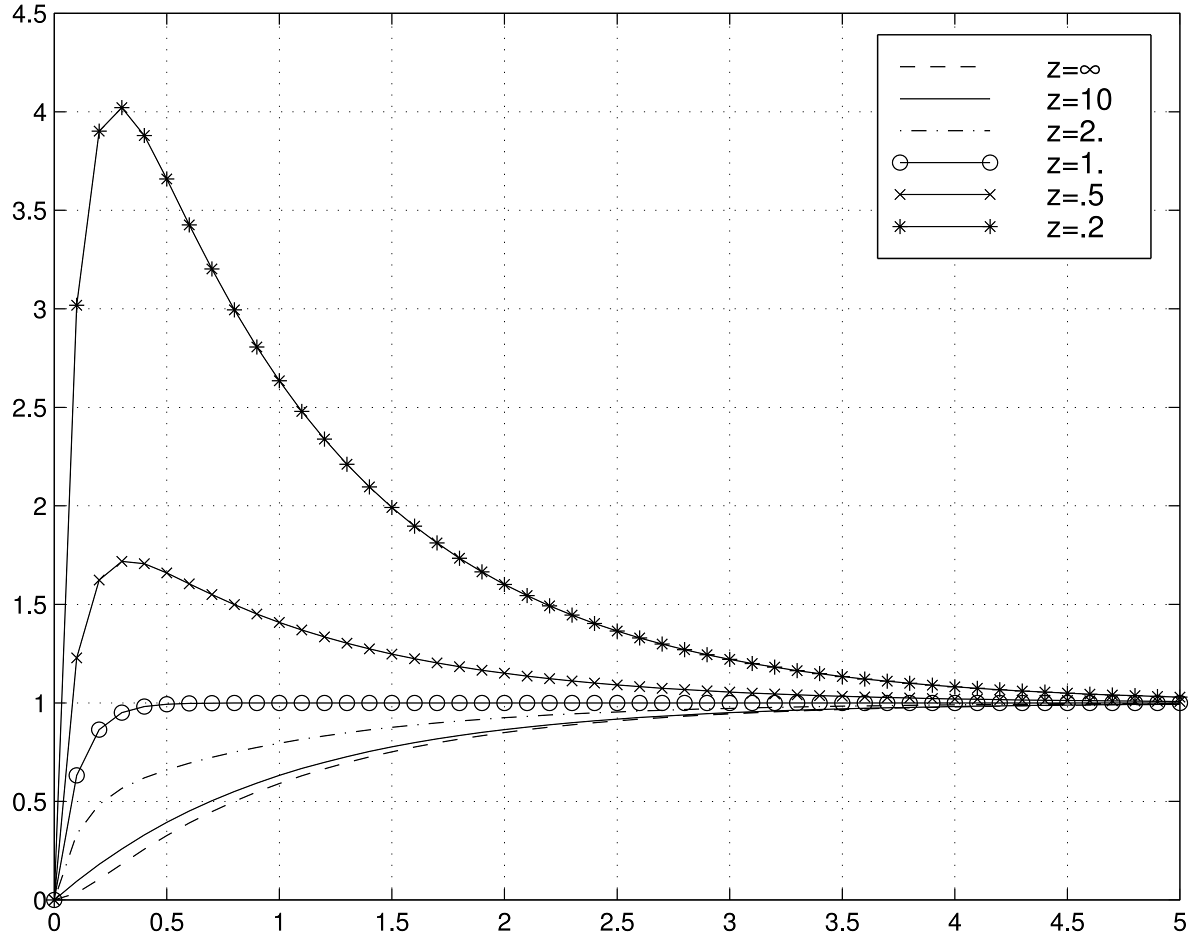
- Consider the following system

$$G(s) = \frac{\frac{1}{z}s + 1}{(s+1)(0.1s+1)}$$

$$Y(s) = \frac{1}{s} - \frac{\frac{10}{9}\frac{z-1}{z}}{s+1} + \frac{\frac{1}{9}\frac{z-10}{z}}{s+10} \qquad y(t) = 1 - \frac{10}{9}\frac{z-1}{z}e^{-t} + \frac{1}{9}\frac{z-10}{z}e^{-10t}$$

Example

- Step response for different values of z



Example

- z >> 10 The pole at s = -1 remains dominant

$$\frac{10}{9}\frac{z-1}{z} \approx \frac{10}{9} \qquad \frac{1}{9}\frac{z-10}{z} \approx \frac{1}{9} \qquad y(t) = 1 - \frac{10}{9}e^{-t}$$

- z = 10 The zero cancels the pole at s = -10. The system becomes 1^{st} order
- 1 < z < 10 The additional zero speeds up the system
- z = 1 The zero cancels the pole at s = -1.
- z < 1 The additional zero becomes dominant. It speeds up the system and creates an overshoot.

- The general effect of the zero in the left-half plane is to increase the speed of the response. When the zero becomes dominant, an overshoot occurs.

Higher order systems

- For a stable system, the relative magnitudes of the residues determine the relative importance of the corresponding poles.
- A pair of closely located poles and zeros will effectively cancel each other.
- If a pole is located very far from origin, the residue of this pole may be small and its response will last for a short time.
- Pole having very small residues contribute little to the transient response and correspondingly may be neglected
- After neglecting, the higher order system may be approximated by a lower order one.